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RATIONAL SIMPLIFICATIONS FOR THE BUCKLING LENGTH OF COLUMNS

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Introduction

The following paper summarizes one of the most extensive and, from a practical standpoint, important structural research efforts of the past decade. During this period it has engaged not only my own attention but also that of numerous other investigators. This problem is the determination of the correct theoretical basis for a rational statement of effective length of columns for use by the practical structural design engineer.

Elementary theory teaches that a pin-ended column has an effective length equal to its actual length, but such idealization is not necessarily applicable in practice because of the varied end restraints present. The dilemma may be most readily recognized in the light of the correct statement that, theoretically at least, it is possible for the *effective length* of a column to vary anywhere from zero to infinity times the actual column length! It is awesome that a subject having so fundamental an effect on both the safety and the economy of the structure has been so blandly ignored in specifications in the past.

As in all modern research endeavors, the efforts to solve this problem have been carried out by many hands and minds functioning in some respects as a "team," yet in other respects with a large degree of independence and even overlapping work. The common bond has been the Column Research Council of the Engineering Foundation, New York, N. Y., under whose aegis the results have been gathered for eventual condensation and approval. Prominent investigators who have worked on various phases of the problem include F. Bleich, H. Bleich, Hoff, Winter, Newmark, Slavin, and Perri, as well as others (see references), and I respectfully acknowledge their accomplishments.

The specific problem discussed in this monograph is the determination of the *effective length* of a compression member for use in design or analysis. In general, the presentation proceeds from the more rigorous methods to the various simplifications or approximations.

Fundamental Considerations

Compression member with end restraints. The behavior of a compression member, which in practice is almost universally part of a

structural framework, is determined by the several restraints imposed by the framework upon the ends of that member. It is perhaps fortunate that one is thus able to focus attention immediately upon the individual compression member with its so-called end conditions imposed by the framework, as designers normally do concentrate on the design of the individual members rather than on that of the total framework. Even where analysis, rather than design, is involved, as in rating a given column in a bridge for permissible load, it is more convenient to deal with the individual column and its restraints than with the framework as a whole.

Rigorous and tested methods of calculating the stability of compression members that are part of rigid jointed frameworks are available.[1-14] These methods involve the application of any of the classic or modern techniques of indeterminate analysis modified to take account of the effect of axial loads and of plastic action on the stiffness or rigidity of the member. However, these procedures are far too complex for routine design, and they are not directly applicable to it, except through the equivalence of these procedures to the determination or approximation of end restraints on the component members of the framework.

End restraints. These end restraints are of various types, may occur in several directions, and also may vary in magnitude as the axial loads in the framework change. The restraints may be flexural or torsional, comprising a rotational restraint or rigidity offered by the framework against the angular rotation of the ends of the compression member in either or all of the three normal planes at each end. Alternately, the restraints may be translational, sometimes called directional, involving resistance by the framework against linear movement, that is, "deflection" or "translation," of the ends of the compression member in three normal directions at each end. In general, these two types of restraints may be thought of as analogous to springs, either rotational like the mainspring of a watch, or translational like a spring balance. The numerical magnitude of the restraints or the corresponding "spring constants," which are measured in typical units as foot-pounds per radian, or as pounds per inch, respectively, need not be, and actually are not, constants, but vary with varying loads. The variation is further complicated by nonlinear stress strain relations in the plastic range.

American engineers familiar with moment distribution will immediately recognize that a flexural restraint against rotation of one end of a compression member offered by an adjacent member in the same plane is the same as the stiffness of the adjacent member. The absolute value of this—where no axial load exists in the restraining member—is known

to be the following for the two limiting conditions at the far end, as shown in FIGURE 1a and b

$$K = \frac{4EI}{L} \tag{1a}$$

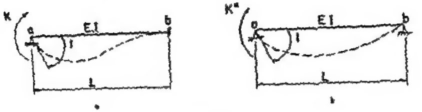

FIGURE 1

$$K'' = \frac{3EI}{L} \tag{1b}$$

To modify for effects of axial load, that is, tension or compression, in the restraining member, these values[1] become, respectively

$$K = 4S\frac{E_t I}{L} \tag{1c}$$

$$K'' = 4S''\frac{E_t I}{L} \tag{1d}$$

where E_t is the tangent modulus of elasticity, and S and S'' are values taken from the graph of FIGURE 2[15] or from extended tabular values.[16]

Engineers working with moment distribution on rectangular frames are also familiar with the concept that a translational restraint offered by a member against horizontal movement of its end, when no axial loads are present, has either of the following values of limiting translational stiffness of that member, as shown in FIGURE 3a and b

$$T = 12\frac{EI}{L^3} \tag{2a}$$

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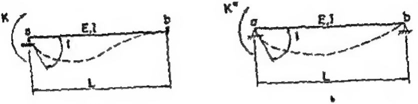

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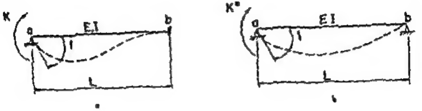

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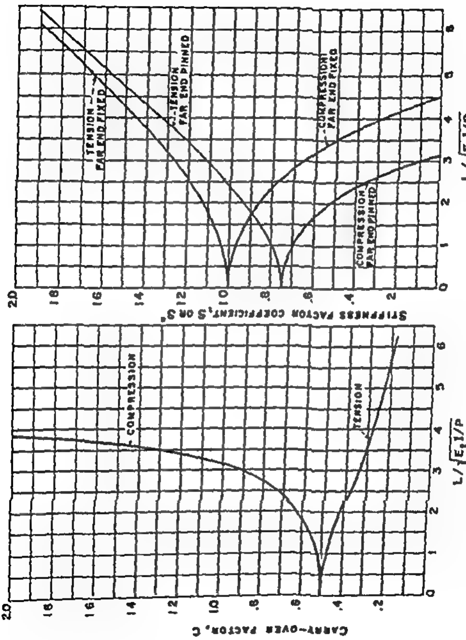

FIGURE 2. Stiffness and carry-over factors with axial load.

To modify for effects of axial load, that is, tension or compression, in the member, these values[6] become, respectively

$$T = \frac{E_t I}{L^3}\left[8S(1+C) - \frac{PL^2}{E_t I}\right] \tag{2c}$$

$$T'' = \frac{E_t I}{L^3}\left[4S'' - \frac{PL^2}{E_t I}\right] \tag{2d}$$

where E_t is the tangent modulus of elasticity, P is the absolute value of the load in the member, and S, S", and C are taken from the graphs of FIGURE 2 or from the corresponding published tabular values.

Torsional rotational restraints need not be discussed here, since their effects may be disregarded for ordinary frameworks.

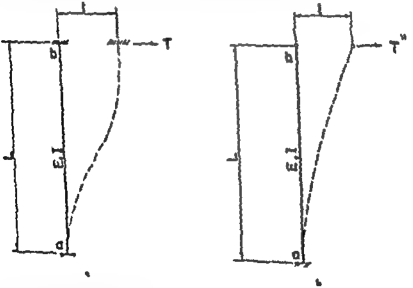

FIGURE 3

Simplification of frameworks The stability of the compression member has been considered above in terms of a member in a general space framework However, it is convenient to break down the structure into plane frameworks, and to consider the buckling of the member in terms of its buckling within the plane of the main framework of which

it is a part, for example, in a main bridge truss, and again as buckling normal to the main framework, which is sometimes spoken of as out of the plane, for example, that which takes place in the plane of the horizontal bracing of a bridge truss. Buckling of the compression member due to torsional restraints—against twisting of the member—at its ends is not considered here, but may be taken into account whenever necessary; in most conventional frames it is of no consequence, particularly since restraints against twisting are usually very large.

Types of buckling. The compression member in a plane framework may be characterized as to the type of buckling failure to which it is subjected as follows (1) that in which the translational rigidity at the ends of the member is so great that little or no translation takes place, a type of action usually found in truss frameworks; (2) that in which both translation and rotation of the ends of the member are possible, a type of action usually found in open rectangular rigid frames; and (3) that in which rotational rigidity at the ends of the member is so great that little or no rotation takes place, for example, a rectangular rigid frame with extremely heavy or stiff upper horizontal cross members. This third type is usually grouped with the second type, leaving two simple general categories: buckling with rotation only, and buckling with rotation and/or translation.

Special cases of the above would include the conditions wherein either the translational or rotational rigidity, or both, at one or both ends of the compression member is zero. One example would be a rectangular rigid frame with vertical columns fixed at their bases, but hinged at their tops.

A classification of the type of buckling is extremely important for designing compression members because the presence of sideway (translation) has a major and often overwhelming effect for certain types of frameworks (nontriangular).

Limitations. The treatment below is altogether general and applies to steel, aluminum alloys, and other structural metals. In certain instances, notably with structural steel, important simplifications are possible because of the relatively rapid transition between elastic and plastic action, and these will be cited as they occur.

The theoretical bases of the buckling action of a column in a plane framework in which the members are subject to axial loads only have been well established by tests and by usage. Recognition of this is indicated by the inclusion of provisions for this type of buckling action in the official specifications of countries such as Great Britain and Germany.[17,18]

It is sufficient here to discuss those frameworks in which axial load effects predominate. Engineers familiar with stability problems

will recognize in this treatment the solution of an indeterminate type of system by energy methods or by other analytic procedures leading to a system of linear simultaneous equations the vanishing of the determinant of which furnishes the stability criterion, or by convergence methods, such as moment distribution or relaxation, in which stability is tested by a simple physical concept.

The effects of primary or secondary bending moments already present in many frameworks at the instant the system passes from stable to unstable equilibrium are omitted in this discussion. In general, these problems differ from those previously mentioned in that the concept of stress as a criterion for stability is introduced.

It must be noted that the findings with respect to frameworks subject to axial loads apply to loads that produce elastic or plastic stresses in the compression member or in its restraining members of the framework. This is possible as long as the concept of an effective or reduced modulus is used. The tangent modulus is accepted as the effective modulus both for tension and for compression members. The following treatment applies to the buckling of compression members that are straight, homogeneous and of constant cross section.

Effective length. Once the end restraints on a compression member of length L are calculated or approximated in magnitude, they determine the value of the effective length kL to be used in the design column formula. An example of such design formula is the general Euler formula:

$$P = \frac{\pi^2 E_t I}{(kL)^2} \quad \text{or} \quad \sigma = \frac{\pi^2 E_t}{(kL/r)^2} \tag{3}$$

in which P is the critical or buckling load, σ is the critical or buckling average stress, E_t is the effective modulus—the tangent modulus, I is the moment of inertia of the cross section of the member, r is the radius of gyration of the cross section of the member, L is the length of the member; and k is the factor by which the actual length must be multiplied to obtain the reduced length. For certain theoretical end conditions, k is known and may be inserted *immediately* in EQUATION 3 without further calculation, as shown in FIGURE 4.

For actual columns in structures wherein infinite rigidity corresponding to fixed ends, or zero rigidity corresponding to hinges, or complete freedom to translate corresponding to a free end, are not encountered, the k values may vary between or be greater than any of those shown in FIGURE 4. For example, in a rigid frame with hinged bases, as shown in FIGURE 5, the value of k may actually approach infinity as the flexural rigidity of the horizontal member decreases toward zero. Thus, in actual

structures it is necessary to apply a procedure for determining k more in line with the actual magnitudes and types of restraints existing

The effective length factor k can be determined from the solution of the appropriate equation for the buckling of a compression member with known end restraints, provided the magnitude of these restraints is known or can be approximated

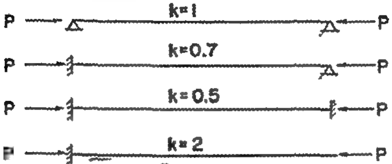

FIGURE 4

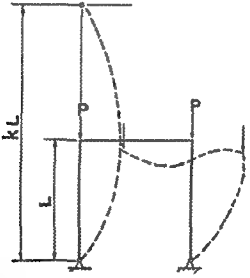

FIGURE 5

In the instance of the buckling of a compression member with known rotational (flexural) restraints at the ends, but with no translation[1] as shown in FIGURE 6,

$$(R_a + R_b) + \frac{R_a R_b}{K} + K^{*} = 0 \tag{4}$$

where R_a and R_b are the rotational restraints in foot pounds per radian at ends a and b respectively, K is the absolute rotational stiffness of the member, the far end fixed, and with no translation at either end

(EQUATION 1c), and K'' = the absolute rotational stiffness of the member, the far end pinned, and with no translation at either end (EQUATION 1d). Both K and K'' are trigonometric functions of the applied load P, per EQUATIONS 1c and 1d, as previously stated.

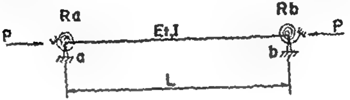

FIGURE 6

To design a column for a given load and end restraints by means of this equation, one would select a trial cross section, calculating the area and moment of inertia I. The average stress, then, P/a, would enable determination of E_t, and the above equation would be tested. The cross section would be modified by trial until the equation was satisfied.

In the instance of the buckling of a compression member with known rotational and translational restraints at both ends (FIGURE 7), and in which the two translational restraints may be replaced by an equivalent

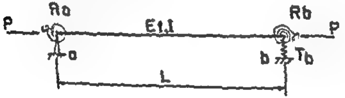

FIGURE 7

at one end only, EQUATION 4 also applies. The exception in this application is that K and K'' are replaced by the more complex expressions $\bar{K}$ and $\bar{K}''$, representing rotational stiffnesses of the member with the far end fixed and pinned, respectively, but with the near end translationally restrained by a linear spring of stiffness T_b pounds per inch, as follows[6]

$$\bar{K} = K\left[\frac{(1-C)+r(1+C)}{2}\right] \tag{5a}$$

$$\bar{K}'' = K''\left[\frac{2r}{r(1+C)+(1-C)}\right] \tag{5b}$$

structures it is necessary to apply a procedure for determining k more in line with the actual magnitudes and types of restraints existing.

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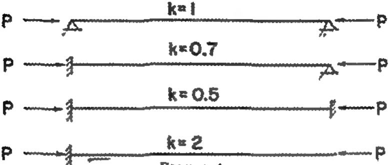

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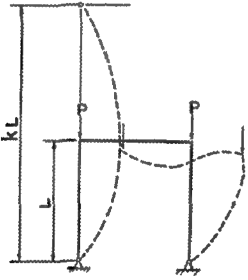

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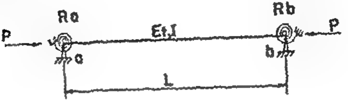

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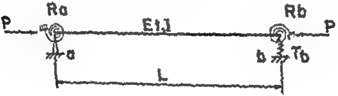

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$$\bar{K}'' = K''\left[\frac{2r}{r(1+C) + (1-C)}\right] \tag{5b}$$

where $t=(T_b - P/L)/(T_b + T)$, T_b is the translational restraint at b, T is the translational stiffness of ab (as shown in EQUATION 2c), and C is the carry-over factor, from the charts of FIGURE 2. The application to design again would be trial and error, with repeated modification of a trial cross section until the equation was satisfied

Approximation of end restraints Both my own investigations and those of others indicate that the end restraints on compression members of frames of type 1 without translation, such as trusses, can be approximated satisfactorily and without impairing either the safety or economy of the structure by considering each compression member as being restrained rotationally only by the "adjacent" tension and minor compression members coming into its ends Thus, in a bridge truss, a top chord compression member may be considered as being restrained rotationally by the diagonal and vertical members coming into its ends and, if the far ends of the adjacent members are considered as pinned, a conservative result will be achieved Such approximation thus neglects the effects of restraining members far removed from the compression member, and which, by extension of the St. Venant principle, have small effect on the member In general, such "three-bay groups" with the compression member in the central bay form the basis for approximation of end rotational restraints, where no translation is involved, throughout the remainder of this paper

When two highly stressed compressive members, for example, AB and BC in FIGURE 8, are restrained by a common restraining member BE, the restraint of BE should be divided in design between the two compressive members In analysis, the proration of the restraint cannot be made arbitrarily, it must produce critical loads simultaneously in the two compression members

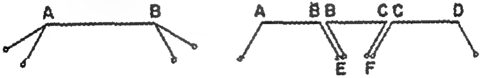


FIGURE 8

Chart solutions The solution of equations of the type of EQUATION 4 for the buckling of restrained columns with rotational restraints only in structures of type 1, as previously defined, is greatly simplified by the use of charts and nomograms, many of which have been prepared for this purpose. FIGURES 9 and 10 are typical illustrations

One approximation of the graphic solution that yields acceptable accuracy is the use of only a single-line chart for the buckling of a

member ab with equal rotational restraints at each end, and the computation of the buckling load of a member ab with unequal restraints as equal to the geometric mean of the buckling loads of the member calculated first, with symmetrical restraints R_a and second, with symmetrical restraints R_b.

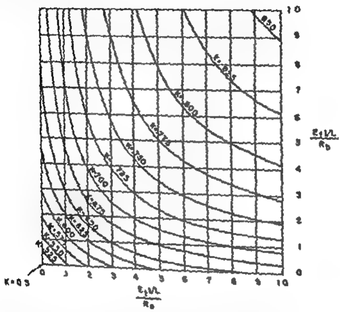


FIGURE 9. Effective length factors k for columns with known rotational restraints R_a and R_b when there is no translation. Reproduced by permission [illegible] the *Journal of the Royal Aeronautical Society*.[10]

A very convenient simplification of the solution of EQUATION 4 is also possible by use of the approximation of Newmark[19] for end fixity from the known rotational restraints, the error being less than 4 per cent, as follows

$$\frac{1}{k^2} \simeq \left(\frac{\pi^2 + 4n_a}{\pi^2 + 2n_a}\right)\left(\frac{\pi^2 + 4n_b}{\pi^2 + 2n_b}\right) \tag{6}$$

where $n_a = R_a/(E_tI/L)$, $n_b = R_b/(E_tI/L)$, and R_a and R_b are end rotational restraints.

A graphic solution for the buckling of a compression member elastically restrained by both flexural—rotational—restraints and by translational restraints in structures of type 2, as defined above, is given in FIGURE 11. In general, this buckling condition is difficult, not only

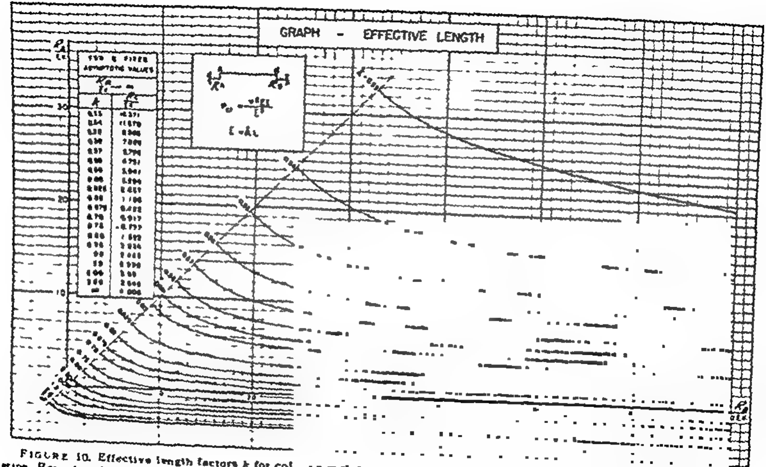


FIGURE 10. Effective length factors k for colu ns with known rotational restraints R_a and R_b when there is no translation. Reproduced by permission of the Cornell University Engineering and Experimental Station Bulletin No. 36.[12]

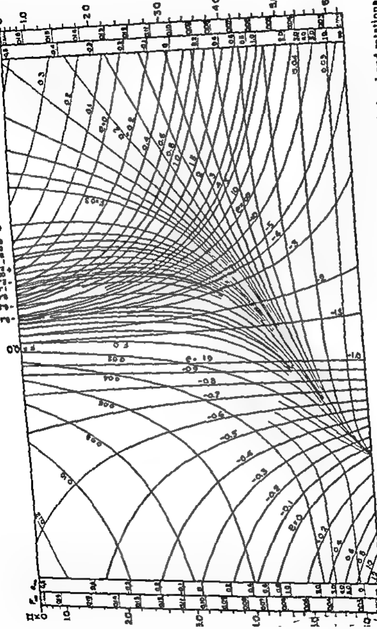


FIGURE 11. Chart for determining effective length factor k for columns with known translational and rotational end restraints. Reproduced by permission of *Bauingenieur, Springer Verlag, Berlin, Germany.*

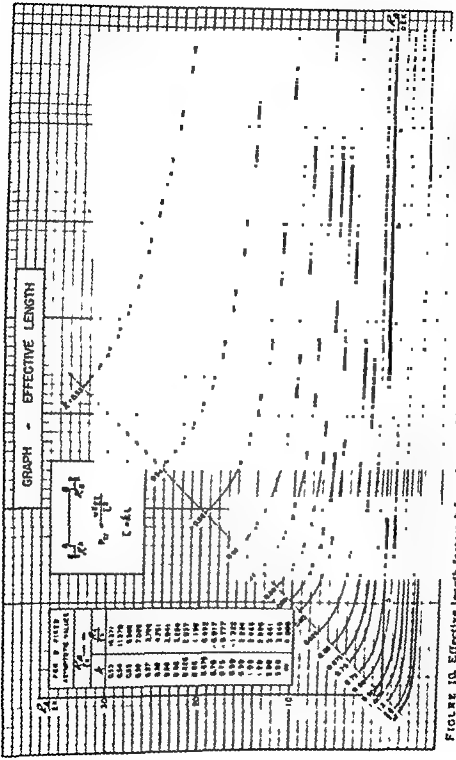


FIGURE 10. Effective length factors k for columns with known rotational restraints R_a and R_b when there is no translation. Reproduced by permission of the Cornell University Engineering and Experimental Station Bulletin No. 36.12

Specific Applications

(1) *Frameworks with little or no translation of joints* In triangulated truss frameworks, loads are usually applied to the joints themselves, producing axial loads in the members if the joints are hinged. Deflections of the members normal to their line of action are relatively small and are due to the axial deformations of the members under load. If the joints are rigidly connected, for instance, by welding or by using heavily riveted gusset plates, some secondary bending is induced The effect of these secondary distortions on the buckling of the truss framework is usually small and may be neglected in the buckling analysis. This simplifies the problem considerably, reducing it to that *of a column* elastically restrained against rotation at both ends.

Chords of trusses The simplest, although conservative, recommendation possible with respect to compression chords of truss frameworks is that the effective length of each member against buckling within the plane of the truss be taken as the theoretical length L, the distance between panel points, that is, $k=1$ (FIGURE 13)

Such a generalization disregards the restraints offered by web members or by the adjacent compression chord members. In the case of trusses with a fixed live loading, such as roof trusses, the neglect of the restraint offered by adjacent compression chord members to each other is justifiable because all such members would approach their buckling load simultaneously as the loads are increased, and are therefore incapable of offering restraint In the case of trusses with moving live loads, such as bridge trusses, such justification may not be strictly valid for all truss configurations, nevertheless, for the usual patterns of trusses there *is* sufficient similarity of loading producing maximum chord stresses in all chords to warrant acceptance of this generalization

When the junctions of the chain of n compression chords at the truss supports are rigidly connected with the tension chords, the following formula,[7] in which n is the number of compression chords, satisfactorily takes account of the additional restraint thereby provided if the chords have constant cross section (FIGURE 14)

$$k=\sqrt{1-\frac{5}{4n}} \tag{7}$$

As indicated previously, the above assumptions neglect restraints afforded by tension and lightly loaded compression diagonals, and therefore result in a conservative solution. To take advantage of these actual restraints, the more general solution of the compression member with rotational end restraints—EQUATION 4, or graphical solutions—should be resorted to

to simplify, but also to generalize except for specific types of structures such as rigid frames. For this reason solutions for various types of such structures are given in greater detail further on in this paper.

An application of the chart (FIGURE 11) can be made with FIGURE 12. Let $o_a = R_a/E_t I/L$; $o_b = R_b/E_t I/L$; $t_a = T_a/E_t I/L$; $t_b = T_b/E_t I/L$; and $F = 1/t_b - 1/t_a$. Lay in the curve that bisects those curves representing given values of o_a and o_b, determine the intersection of this "middle" curve with the curve for given F; read horizontally in the right or left margin the value of π/k. For a multiple solution, choose the lowest

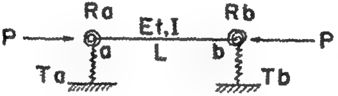


FIGURE 12

value of π/k. For example if $o_a = 10.28$, $o_b = 27.40$, and $F = \infty$, then read $\pi/k = 3.03$, $k = 1.04$. The columns o_∞ and F_∞ give the location of asymptotes to the o and F curves and are of interest in special cases, such as $o_a = o_b$.

Recapitulation. The treatment thus far has been altogether general, and although this is satisfactory for analytical purposes, it is perhaps too involved for specific use, except on a broad basis. It must be understood that EQUATION 3, that is, the Euler formula, and equations of the form of EQUATION 4 covering the buckling of elastically restrained members are both fundamentally identical. In EQUATION 4, however, one deals directly with the numerical values of the restraints, and although the length factor k enters through substitution of its value from EQUATION 3, the fundamental need for an evaluation of k and its subsequent use in EQUATION 3 for design has been obviated since, by the procedure thus far presented, the design takes place as part of the solution for EQUATION 4. All axial loads considered in these equations are ultimate loads, and the analysis becomes one of ultimate rather than of working loads.

In general, therefore, and especially since designers are accustomed to working with column formulas like that in EQUATION 3, it appears preferable to utilize alternate procedures that give specific values for k for certain classes of members and configurations, for later use with a column formula.

members may be designed for a reduced length kL equal to 8 times the actual length

For the case of crossed diagonal members of trusses (FIGURE 16), the values of effective length factor k given in TABLE 3 may be em-

TABLE 1

EFFECTIVE LENGTH FACTORS k FOR DIAGONALS*

t	o	$u = 0$	0.05	0.10	0.15	0.20	0.25	0.30
$t = 0.3$	0.2	0.52	0.53	0.55	0.56	0.57	0.58	0.59
	0.4	0.54	0.55	0.57	0.58	0.59	0.60	0.61
	0.6	0.56	0.57	0.58	0.59	0.60	0.61	0.62
	0.8	0.57	0.58	0.59	0.60	0.61	0.62	0.63
	1.0	0.58	0.59	0.60	0.61	0.62	0.63	0.64
$t = 0.4$	0.2	0.53	0.54	0.55	0.56	0.57	0.58	0.59
	0.4	0.55	0.56	0.57	0.58	0.59	0.60	0.61
	0.6	0.57	0.58	0.59	0.60	0.61	0.62	0.63
	0.8	0.58	0.59	0.60	0.61	0.62	0.63	0.64
	1.0	0.59	0.60	0.62	0.63	0.64	0.64	0.65
$t = 0.5$	0.2	0.56	0.57	0.57	0.58	0.59	0.60	0.60
	0.4	0.58	0.59	0.59	0.60	0.61	0.62	0.62
	0.6	0.59	0.60	0.61	0.62	0.63	0.64	0.64
	0.8	0.60	0.61	0.62	0.63	0.64	0.65	0.66
	1.0	0.61	0.62	0.63	0.64	0.65	0.66	0.67
$t = 0.6$	0.2	0.61	0.62	0.62	0.63	0.63	0.64	0.64
	0.4	0.63	0.63	0.64	0.65	0.65	0.66	0.66
	0.6	0.64	0.64	0.65	0.66	0.66	0.67	0.67
	0.8	0.64	0.65	0.66	0.67	0.67	0.68	0.69
	1.0	0.65	0.66	0.66	0.67	0.68	0.69	0.69
$t = 0.7$	0.2	0.70	0.70	0.70	0.70	0.71	0.71	0.71
	0.4	0.70	0.70	0.71	0.71	0.71	0.72	0.72
	0.6	0.70	0.70	0.71	0.71	0.72	0.72	0.73
	0.8	0.70	0.70	0.71	0.72	0.72	0.73	0.73
	1.0	0.70	0.71	0.71	0.72	0.73	0.73	0.74

$$o = \frac{L_C\, I_D}{L_D\, I_C}$$

$$u = \frac{1}{3}\,\frac{L_T\, I_D}{L_D\, I_T}$$

$$t = \frac{L_C}{L_D}\sqrt{\frac{P_C I_D E_{tD}}{P_D I_C E_{tC}}}$$

*Reproduced by permission from *Theorie und Berechnung der Eisernen Brücken*.[8]

The above simplifications can also be applied to the determination of effective length of chords against buckling normal to the truss and in cases in which the chord becomes a component of, for example, an upper horizontal bracing truss system. Where a bracing system is lacking as in the case of pony trusses, the above approximations do not apply, and one must resort to procedures recommended by the researchers investigating this type of action for the Column Research Council.

FIGURE 13

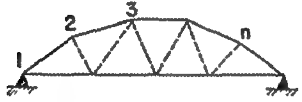


FIGURE 14

Web members. If restraints of tension diagonals are disregarded, compression diagonals may be considered as flexurally restrained against buckling in the plane of the truss by adjacent upper and lower chord members (FIGURE 15), yielding the solutions for k based on certain simplifying assumptions given in TABLE 1.

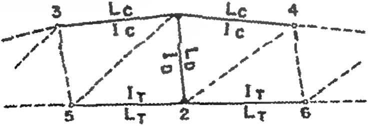


FIGURE 15

A simpler and more conservative table[8] (TABLE 2) results from neglecting the restraints offered the web members by the compression chords, and by assuming equality of effective modulus for the diagonal and lower chords. A resultant more general approximation is that web

TABLE 4

EFFECTIVE LENGTH FACTORS—K-TRUSS WEBS

P_2/P_1	k
0	0.73
0.2	0.67
0.4	0.62
0.6	0.57
0.8	0.53
≥1.0	0.50

axial loads in the chords, for example, in bridges and crane runways, the chords will offer considerable restraint, and the approximations of TABLE 1 appear justifiable. If the live loading on the structure is fairly stationary, producing maximum loads more or less simultaneously in all members, as in the case of roof trusses, then the restraints offered to compression diagonals by the compression chords will be zero. A proper simplification of the latter would be to design all such web members for the k values given in TABLE 2.

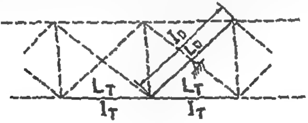


FIGURE 16

For buckling perpendicular to the plane of the main truss, the diagonals should be designed for $k=1$, unless more detailed information about the make up of the cross frames (perpendicular to the truss) is available. For example, given cross frames of Type 1 (FIGURE 18), and assuming that no translation of joint is possible, it appears satisfactory to design for $k=0.8$ while for Type 2 (FIGURE 18) it appears satisfactory to use $k=0.7$. Where translation of the cross frames is possible, more exact analysis by methods described in a later section should be undertaken.

Effects of materials and the factor of safety. Effective length design criteria are markedly affected by the type of material involved and by the varying requirements set by the specifications regarding the factor of safety against failure.

ployed. A more general approximation of the above is to use a reduced length kl equal to 0.5 times the actual length of the diagonals.

For the case of the vertical web members in K-trusses, with buckling erpendicular to the plane of the truss (FIGURE 17), the approximations r k given in TABLE 4 should be employed.[7]

TABLE 2

WEB MEMBER APPROXIMATIONS FOR k

$\frac{1}{6}\frac{L_T}{L_D}\frac{I_D}{I_T}$	k
0	0.700
0.02	0.714
0.04	0.724
0.06	0.738
0.08	0.749
0.10	0.760
0.12	0.770
0.14	0.779
0.16	0.788
0.18	0.797
0.20	0.805

TABLE 3

EFFECTIVE LENGTH FACTORS—CROSSED DIAGONALS

$\frac{2}{3}\frac{\left(\frac{EI}{L}\right)_T}{\left(\frac{EI}{L}\right)_D}$	k
0	0.437
0.2	0.449
0.4	0.457
0.6	0.463
0.8	0.467
1.0	0.471
1.2	0.474
1.4	0.476
1.6	0.478

In general, the above approximations for web members apply to trusses with fairly uniform spacing of panels and without unusual configurations. If the web members receive their maximum axial load under a moving live load system that does not simultaneously produce maximum

restraints even at high loads. Even with such small slenderness ratios as 50, there is a sizable difference between the buckling load of the pinned and fixed column, and the use of the proper k value may have a significant effect even for these low slenderness ratios.

With most steel specifications a higher allowable stress is provided in tension than in compression, expressing the need for higher factors of safety for the latter type of action. Since tension members, if designed *only* by this criterion, would reach their yield point long before the compression members, the major restraints offered to the latter disappear Under conditions where, for example, factors of safety in the range of 1 7 in tension and 2 2 in compression are employed, it appears desirable to design *all* members for $k=1$, unless one incorporates in the restraining tension members an excess of area for reasons other than to meet stress needs, such as minimum L/r requirements and the inability to secure commercial sizes exactly equal to those theoretically *required*.

Some generalizations are possible for the case of design for moving loads in steel by assuming uniform live and dead loadings. Assuming factors of safety of 2 25 in compression and 1.83 in tension, it may be shown that, when the dead load is zero, only those web members within the middle six tenths of the span can possibly have end *restraints* from the tensile chords, and that this range decreases with *increasing* dead load until the dead load reaches about 1 68 times the live load, beyond which point the tensile chords can be counted upon to offer no restraints at all

It is apparent then that for steel roof trusses, and possibly some lightly loaded highway trusses, it is quite possible that with present factors of safety, very little if any restraint will be offered by the tension chords to the web members, even in part span loadings, unless one resorts to overdesign

Along the same lines, although *compression* chords offer restraints to web members in part span loadings, with present factors of safety such restraints alone are not sufficient to reduce the k values beyond 0 88 A combination with considerable tensile restraint would be necessary to bring the k values down to the 0 8 value given in TABLE 1, suggested by Bleich[3] for web-member design.

(2) Frameworks with translation of joints Where translation of joints, that is, sideway or lateral movement, is possible, as in the case of rigid frames, the *effective* length of columns may generally be considerably larger than their actual length This is illustrated in FIGURE 19

Similarly, where free-standing columns occur, such as those supporting outdoor crane runway girders and similar structures, a k value [illegible] en previously noted as appropriate It has been further noted

With material having a well-defined yield point, such as structural steel, there is a marked and rapid deterioration in the effective modulus E_t in the region of the yield point, beyond which E_t attains the value of zero. As a result of this, both compression members and tension members lose their ability to offer restraints to other compression members, and the effective length factor is reduced to unity.

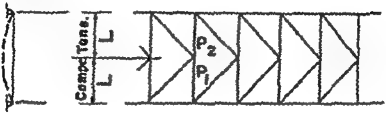


FIGURE 17

Moreover, the deterioration of the effective modulus of structural steel is so great for columns with slenderness ratios less than about 60 that the critical buckling load for a fixed column is only about 1 or 2 per cent different from that with pinned ends, assuming no translation Thus, the introduction of k values has little if any practical effect on the design of these columns Similar remarks apply also to low-alloy and silicon steels. The difference in buckling loads of steel columns for the pinned and fixed-end conditions increases steadily as the slenderness ratio rises from 60 to 125, at which latter point the fixed ended column load may approach 100 per cent more than that of the pin ended column. The use of appropriate k values thus can result in appreciable economy in steel design for this slenderness ratio range

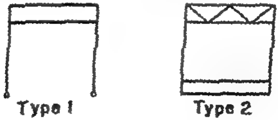


FIGURE 18

All of these remarks apply only to the column without translation, since translation has a significant effect, even for lower slenderness ratios

With such materials as aluminum and magnesium alloys the reduction in E_t, while large as the yield stress is approached, still remains finite, and the adjacent members retain some increase of capacity for offering

[illegible]

of k [illegible]
fixed ended symmetrical frame solution (FIGURE [illegible]) [illegible]

A more extended chart tabulation of k values for one-story frames as derived from the various cases given in the German specifications is presented in FIGURE 24.[21] Another case for which an exact solution is known is the two-story bent[7] shown in FIGURE 25.

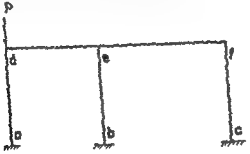


FIGURE 20

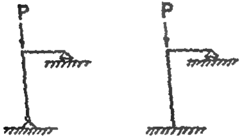


FIGURE 21

For multistory rectangular frames, approximation formulas of the type given by Bleich[7] may be employed as a guide to the value of k, but the assumptions under which these approximations are derived do not usually satisfy actual conditions, and the results may be in error. I do not feel that these solutions are of sufficient general validity to justify their inclusion in specifications.

British standards. Engineers have noted the simplicity of the current British standards[17] for using structural steel in building which include the recommendations given in FIGURE 26 for effective lengths of building

in FIGURES 5 and 19 that, theoretically, for some types of frameworks it is possible for the columns to approach an infinite effective length, although it can hardly be said that such types of structure are of a practical nature.

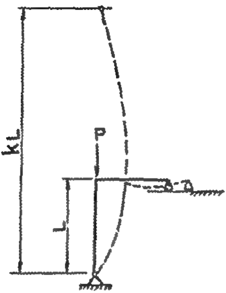


FIGURE 19

Direct application of EQUATION 4 *for a column with end restraints.* EQUATION 4, or its graphic solution, FIGURE 11, covering the buckling load of a column with end rotational and translational restraints, can be applied for certain simple cases in which the end restraints may be readily determined or approximated. Thus, in FIGURE 20 the column translational stiffnesses of *bo* and *cf* can be conservatively stated as if the tops were pinned and added together to yield the translational restraint; similarly, the rotational stiffness of *do* could be estimated conservatively as that with end *o* pinned. The resultant k value from the charts would be conservative in all cases. Similarly, the cases shown in FIGURE 21 are solved simply by EQUATION 4 or its graphic equivalent.

For design purposes, however, the loading conditions of FIGURE 20 are not usually critical because all of the columns may be simultaneously under high stress, and the effects of the axial load would diminish their translational stiffness. Hence, for usual design purposes, analysis by EQUATION 4 is cumbersome, it would be more appropriate to designate critical design k's for specific and common structural frames and loadings. These are covered below.

The solution[7] given in FIGURE 22 may be used for the determination of k values for the specific rigid jointed frame loadings indicated. The fixed ended symmetrical frame solution (FIGURE 23) is also known

A more extended chart tabulation of k values for one-story frames as derived from the various cases given in the German specifications is presented in FIGURE 24 [21] Another case for which an exact solution is known is the two-story bent[7] shown in FIGURE 25

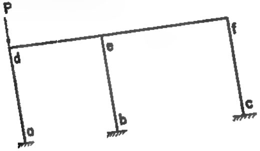


FIGURE 20

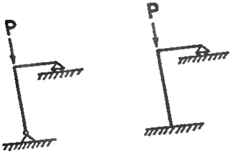
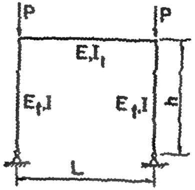


FIGURE 21

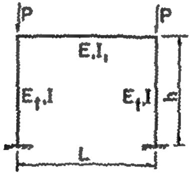
For multistory rectangular frames, approximation formulas of the type given by Bleich[7] may be employed as a guide to the value of k, but the assumptions under which these approximations are derived do not usually satisfy actual conditions, and the results may be in error I do not feel that these solutions are of sufficient general validity to justify their inclusion in specifications

British standards Engineers have noted the simplicity of the current British standards[17] for using structural steel in building which include the recommendations given in FIGURE 26 for effective lengths of building struts



$\dfrac{E_t I / h}{E I_1 / L} =$	0	0.1	0.2	0.5	1.0	3.0	∞
Held against translation $k =$	0.700	0.733	0.761	0.814	0.875	0.963	1.000
Free to translate $k =$	2.00	2.03	2.07	2.17	2.33	3.38	—

FIGURE 22



$\dfrac{E_t I / h}{E I_1 / L} =$	0	0.1	0.2	0.5	1.0	3.0	∞
Held against translation $k =$	0.500	0.524	0.545	0.590	0.626	0.680	0.700
Free to translate $k =$	1.000	1.016	1.030	1.082	1.136	1.302	2.000

FIGURE 23

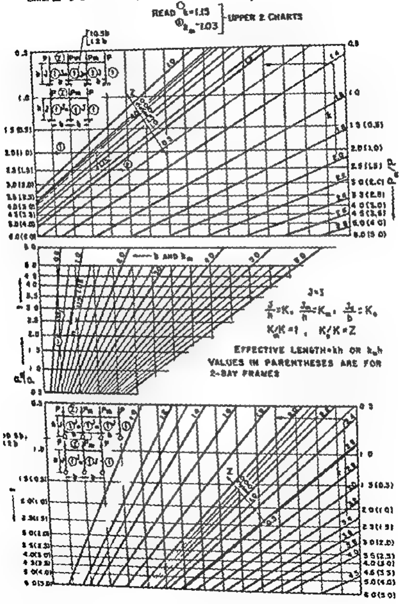
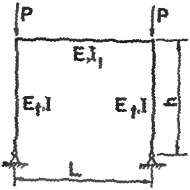
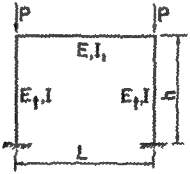


FIGURE 24A. For description, see FIGURE 24B, page 380.



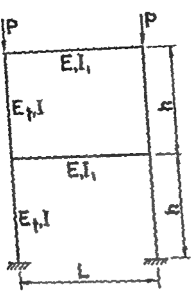
$\dfrac{E_t I / h}{E I_1 / L} =$	0	0.1	0.2	0.5	1.0	5.0	∞
Held against translation $k =$	0.700	0.733	0.761	0.814	0.875	0.963	1.000
Free to translate $k =$	2.00	2.03	2.07	2.17	2.33	3.38	—

FIGURE 22



$\dfrac{E_t I / h}{E I_1 / L} =$	0	0.1	0.2	0.5	1.0	5.0	∞
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
Free to translate $k =$	1.000	1.016	1.030	[illegible]	[illegible]	[illegible]	[illegible]

FIGURE 23



$\dfrac{E_t I / h}{E I_1 / L} =$	0	0.1	0.2	0.5	1.0	2.0	∞
Held against translation $k =$	0.500	0.507	0.668	0.689	0.753	0.803	0.879
Free to translate $k =$	1.000	1.033	1.065	1.160	1.310	1.515	4.000

FIGURE 25

dition between the fifth case and one similar to the second case, but with both ends restrained in the direction shown in FIGURE 27 ($k=1$). Its deficiency lies in the lack of definition of the magnitude of the "partial" restraint, which can produce k values anywhere between the two extremes. While, for small slenderness ratios, the requirement of the k value may not be important, for large slenderness ratios the error in the fourth case may be large.

The British standards do not, of course, cover the important case of FIGURES 5 or 19, where k can become greater than 2. In general, I feel that the British standards represent an extreme of simplification of the more mathematical graph or table of reduced length, and can be improved by more careful definition of restraints as indicated above.

(3) *Arches* The buckling of an arch within its plane represents a special case of framework buckling for certain cases of which solutions are known.[11]

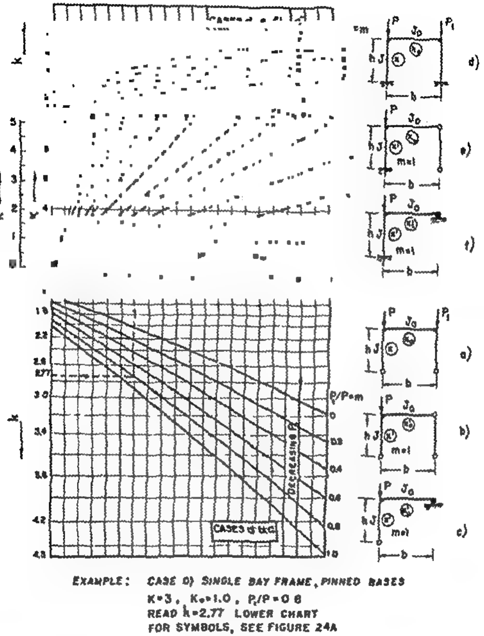


FIGURE 24A and B. *Nomograms for frame buckling per German DIN 4114. Reproduced by permission of Wilhelm Ernst & Sohn.*[21]

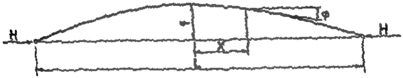
All cases except the fourth deal with either perfect rotational fixity or hinged conditions and with perfect translational fixity or freedom. The first three cases represent conservative approximations of the known solutions. The fifth case is the known solution of the crane column previously mentioned. The fourth case represents an *intermediate* con-

FIGURE 27

For symmetrical parabolic arches with uniform load, the relations for effective length given in FIGURE 28 and EQUATION [illegible] are valid.

$$H_{cr} = \frac{\pi^2 EI\,\text{crown}}{(kL)^2} \qquad (8)$$

These relations are not valid for arches connected with a tie by hangers.



For constant I_x = I crown						
f/L =	0	0.1	0.2	0.3	0.4	0.5
Fixed arch k =	0.350	0.361	0.395	0.454	0.533	—
Two-hinged arch k =	0.500	0.526	0.590	0.713	0.849	1.01
Three-hinged arch k =	0.575	0.588	0.630	0.713	0.849	1.01
For variable I_x = I crown/cos φ						
f/L =	0	0.1	0.2	0.3	0.4	0.5
Fixed arch k =	0.350	0.355	0.373	0.401	0.439	0.485
Two-hinged arch k =	0.500	0.516	0.559	0.627	0.713	0.811
Three-hinged arch k =	0.576	0.579	0.597	0.627	0.713	0.811

FIGURE 28

For buckling normal to the plane of the arch, the *cross section* [illegible] the arch [illegible] subject to translation and to torsional twist. The relationships covering this type of action are a special category of the pony truss buckling problem[18,21,22] and are not cited here.

Type	*Effective length*	
Effectively held in position and restrained in direction at both ends	0.7 L	
Effectively held in position at both ends and restrained in direction at one end	0.85 L	
Effectively held in position at both ends, but not restrained in direction	L	
Effectively held in position and restrained in direction at one end and at the other end partially restrained in direction, but not held in position	1.5 L	
Effectively held in position and restrained in direction at one end, but not held in position or restrained in direction at the other end	 2.0 L	

FIGURE 26

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[illegible] nicklänge von [illegible]

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